Analysis II

Chebyshev Theorem

Subject: Integral Calculus
Educational level: 1st Year University mathematics
Student: Abdeljalil Mostefaoui

Abdeljalil Mostefaoui | Echahid Cheikh Larbi Tebessi university

2025/2026

Copyright © 2026 by **Abdeljalil Mostefaoui**
All rights reserved. No part of this publication may be reproduced, distributed, or transmitted in any form or by any means, including photocopying, recording, or other electronic or mechanical methods, without the prior written permission of the author, except in the case of brief quotations embodied in critical reviews and certain other noncommercial uses permitted by copyright law.
Typeset in LaTeX.

Part I
Introduction to Chebyshev's Theorem

A binomial differential is an integral of the form:

$$I = \int x^m (ax^n + b)^p \, dx$$

where a, b are real constants and m, n, p are rational numbers. According to **Chebyshev's Theorem**, this integral can be expressed in terms of elementary functions *if and only if* one of the following three conditions is satisfied:

1. **Case 1:** p is an integer.
2. **Case 2:** $\frac{m+1}{n}$ is an integer.
   Substitution: $ax^n + b = t^q$, where q is the denominator of p.
3. **Case 3:** $\frac{m+1}{n} + p$ is an integer.
   Substitution: $a + bx^{-n} = t^q$, where q is the denominator of p ($q = \frac{1}{p}$).

Part II
Problem 01

We aim to solve the following integral:

$$I = \int \frac{\sqrt{4x^2 - 9}}{x^3} \, dx$$

Identify Parameters

First, we rewrite the integral in the standard binomial form $x^m(a + bx^n)^p$:

$$I = \int x^{-3}(4x^2 - 9)^{1/2} \, dx$$

Identifying the parameters from the given form:

$$m = -3, \quad n = 2, \quad p = \frac{1}{2}, \quad a = -9, \quad b = 4$$

We assume standard integration constants are implied. We check the condition for rationalizability:

$$\frac{m+1}{n} = \frac{-3+1}{2} = \frac{-2}{2} = -1 \quad (\in \mathbb{Z})$$

Since the result is an integer, **Case 2** applies.

Substitution

According to the method (and noting the denominator of p is 2), we use the substitution:

$$4x^2 - 9 = z^2$$
$$\implies z = \sqrt{4x^2 - 9}$$

Differentiating both sides:

$$8x \, dx = 2z \, dz \implies dx = \frac{z}{4x} \, dz$$

From the substitution, we isolate x^2:

$$x^2 = \frac{z^2+9}{4} \implies x^4 = \frac{(z^2+9)^2}{16}$$

Substituting these into the integral I:

$$\begin{aligned} I &= \int \frac{z}{x^3} \cdot \left(\frac{z}{4x}\, dz\right) \\ &= \frac{1}{4}\int \frac{z^2}{x^4}\, dz \\ &= \frac{1}{4}\int z^2 \cdot \frac{16}{(z^2+9)^2}\, dz \\ &= 4\int \frac{z^2}{(z^2+9)^2}\, dz \end{aligned}$$

Solving the Intermediate Integral

We need to evaluate:

$$\mathcal{I}_z = 4\int \frac{z^2}{(z^2+9)^2}\, dz$$

To solve $\int \frac{z^2}{(z^2+9)^2} dz$, we use **Integration by Parts** on a simpler related integral J. Let:

$$J = \int \frac{1}{z^2+9}\, dz$$

We know that $J = \frac{1}{3}\arctan\left(\frac{z}{3}\right)$. Now, applying Integration by Parts on J:

$$\begin{aligned} u &= \frac{1}{z^2+9} & dv &= dz \\ du &= \frac{-2z}{(z^2+9)^2}\, dz & v &= z \end{aligned}$$

Using the formula $\int u\, dv = uv - \int v\, du$:

$$\begin{aligned} J &= \frac{z}{z^2+9} - \int z \cdot \frac{-2z}{(z^2+9)^2}\, dz \\ J &= \frac{z}{z^2+9} + 2\int \frac{z^2}{(z^2+9)^2}\, dz \end{aligned}$$

Notice that the integral on the right is exactly the term we need for $\mathcal{I}_z$ (divided by 2). Rearranging to solve for the target integral:

$$2\int \frac{z^2}{(z^2+9)^2}\, dz = J - \frac{z}{z^2+9}$$

Multiplying by 2 to match our expression for $\mathcal{I}_z$ (which has a factor of 4):

$$4\int \frac{z^2}{(z^2+9)^2}\, dz = 2\left(J - \frac{z}{z^2+9}\right) = 2J - \frac{2z}{z^2+9}$$

Final Assembly

Substitute $J = \frac{1}{3}\arctan\left(\frac{z}{3}\right)$ back into the expression:

$$I = \frac{2}{3}\arctan\left(\frac{z}{3}\right) - \frac{2z}{z^2+9} + C$$

Finally, return to the original variable x using $z = \sqrt{4x^2-9}$ and $z^2 + 9 = 4x^2$:

$$\begin{aligned} I &= \frac{2}{3}\arctan\left(\frac{\sqrt{4x^2-9}}{3}\right) - \frac{2\sqrt{4x^2-9}}{4x^2} + C \\ &= \frac{2}{3}\arctan\left(\frac{\sqrt{4x^2-9}}{3}\right) - \frac{\sqrt{4x^2-9}}{2x^2} + C \end{aligned}$$

$$\int \frac{\sqrt{4x^2-9}}{x^3}\,dx = \frac{2}{3}\arctan\left(\frac{\sqrt{4x^2-9}}{3}\right) - \frac{\sqrt{4x^2-9}}{2x^2} + C$$

Part III
Problem 02

We aim to solve the following integral:

$$I = \int \frac{\sqrt{4x^2 - 9}}{x^2}\, dx$$

Identify Parameters

First, rewrite the integral in the standard binomial differential form $I = \int x^m (a + bx^n)^p\, dx$:

$$I = \int x^{-2}(4x^2 - 9)^{1/2}\, dx$$

Comparing this to the standard form, we identify the parameters:

- $m = -2$
- $n = 2$
- $p = \frac{1}{2}$
- $a = -9, \quad b = 4$

Checking Chebyshev's Conditions:

- **Case 1:** p is an integer? $\to \frac{1}{2}$ (No).
- **Case 2:** $\frac{m+1}{n}$ is an integer?
$$\frac{-2+1}{2} = -\frac{1}{2} \quad \text{(No)}$$
- **Case 3:** $\frac{m+1}{n} + p$ is an integer?
$$-\frac{1}{2} + \frac{1}{2} = 0 \quad \text{(Yes, Integer)}$$

Since $\frac{m+1}{n} + p$ is an integer, **Case 3 applies**.

Substitution

According to Chebyshev's Theorem for Case 3, we use the substitution $t^q = x^{-n}(a+bx^n)$, or equivalently transforming the term inside the root by factoring out x^2.

$$\sqrt{4x^2 - 9} = \sqrt{x^2(4 - 9x^{-2})} = x\sqrt{4 - \frac{9}{x^2}}$$

Let the substitution be:

$$t^2 = 4 - \frac{9}{x^2}$$

Then, $t = \sqrt{4 - \frac{9}{x^2}} = \frac{\sqrt{4x^2-9}}{x}$.
Differentiating both sides of Eq (2):

$$2t\, dt = -9(-2)x^{-3}\, dx$$

$$2t\, dt = \frac{18}{x^3}\, dx \implies t\, dt = \frac{9}{x^3}\, dx$$

We need to express dx and the integral terms in terms of t. From the substitution $t^2 = 4 - 9x^{-2}$, we have:

$$9x^{-2} = 4 - t^2 \implies \frac{1}{x^2} = \frac{4 - t^2}{9}$$

Solving the Integral

Rewrite the original integral I:

$$I = \int x^{-2}\sqrt{4x^2-9}\,dx = \int \frac{\sqrt{4x^2-9}}{x} \cdot \frac{1}{x}\,dx$$

Notice that $\frac{\sqrt{4x^2-9}}{x} = t$. Also, from the derivative, we can rearrange terms. Let's express strictly in terms of t:

$$I = \int \left(4 - \frac{9}{x^2}\right)^{1/2} x^{-2}\,dx$$

From $t\,dt = 9x^{-3}\,dx$, we have $x^{-3}dx = \frac{1}{9}t\,dt$. Multiply and divide the integrand by x:

$$I = \int x \cdot x^{-3}(4x^2-9)^{1/2}\,dx$$

This path is complex. Let's use the direct values derived:

- $(4x^2-9)^{1/2} = xt$
- $dx = \frac{x^3}{9}t\,dt$

Substituting into $I = \int \frac{(4x^2-9)^{1/2}}{x^2}dx$:

$$I = \int \frac{xt}{x^2} \cdot \left(\frac{x^3}{9}t\,dt\right) = \frac{1}{9}\int x^2t^2\,dt$$

Now substitute x^2 from $x^2 = \frac{9}{4-t^2}$:

$$I = \frac{1}{9}\int \left(\frac{9}{4-t^2}\right)t^2\,dt = \int \frac{t^2}{4-t^2}\,dt$$

Simplifying the Rational Function:

$$\frac{t^2}{4-t^2} = \frac{t^2-4+4}{-(t^2-4)} = -1 + \frac{4}{4-t^2}$$

Thus:

$$I = \int \left(-1 + \frac{4}{4-t^2}\right)dt$$

$$I = -t + 4\int \frac{1}{2^2-t^2}\,dt$$

Using the standard integral $\int \frac{du}{a^2-u^2} = \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right|$:

$$I = -t + 4 \cdot \frac{1}{2(2)}\ln\left|\frac{2+t}{2-t}\right|$$

$$I = -t + \ln\left|\frac{2+t}{2-t}\right| + C$$

Final Substitution

Recall $t = \frac{\sqrt{4x^2-9}}{x}$. Substituting back:

$$I = -\frac{\sqrt{4x^2-9}}{x} + \ln\left|\frac{2+\frac{\sqrt{4x^2-9}}{x}}{2-\frac{\sqrt{4x^2-9}}{x}}\right| + C$$

Simplifying the logarithmic term by multiplying numerator and denominator by x:

$$I = -\frac{\sqrt{4x^2-9}}{x} + \ln\left|\frac{2x+\sqrt{4x^2-9}}{2x-\sqrt{4x^2-9}}\right| + C$$